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Enhanced Numerical Methods for Options with Barriers

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SUMMARY

Most real-world barrier options have no analytic solutions, either because the barrier structure is complex or because of volatility skews in the market. Numerical solutions are a necessity. But options with barriers are notoriously difficult to value numerically on binomial or multinomial trees, or on finite-difference lattices. Their values converge very slowly as the number of tree or lattice levels increase, often requiring unattainably large computing times for even a modest accuracy.

In this paper we analyze the biases implicit in valuing options with barriers on a lattice. We then suggest a method for enhancing the numerical solution of boundary value problems on a lattice that helps to correct these biases. It seems to work well in practice.

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THE PROBLEM WITH BARRIER OPTIONS

As derivative markets have matured, options with barriers[1] have become increasingly popular because of the greater precision with which they allow investors to obtain or avoid exposure. The value of a knockin stock (or index) option depends sensitively on the risk-neutral probability of the stock being in-the-money and beyond the barrier. Similarly, the value of a knockout option depends on the probability of the stock being in-the-money but not beyond the barrier. The analytic solution for these probabilities, and for the value of a European-style knockout option on stock under the standard Black-Scholes assumptions, was published by Merton (1973). This analytic solution provides rapidly computed, accurate values and hedge ratios, so important for managing the risk of large books of exotic and standard options.

Many of the currently traded barrier-style derivatives have no analytic solutions. The analytic method works only for simple barriers at a fixed or exponentially rising level, assuming lognormal stock price evolution and European-style exercise. There are now over-the-counter markets in options whose barriers may have arbitrary time dependence, whose implied volatilities exhibit a skew that corresponds to non-lognormal evolution of the underlying stock price[2], or whose exercise may be American-style. In most of these cases there exists no general analytic solution for the value of the barrier option, and so a numerical solution is unavoidable. The most common numerical techniques involve solving the differential equation on a binomial lattice (Cox, Ross and Rubinstein 1979), using more general (explicit or implicit) finite difference methods, or using the Monte Carlo method of integral evaluation (Boyle 1977). The numerical accuracy of these methods becomes an important issue.

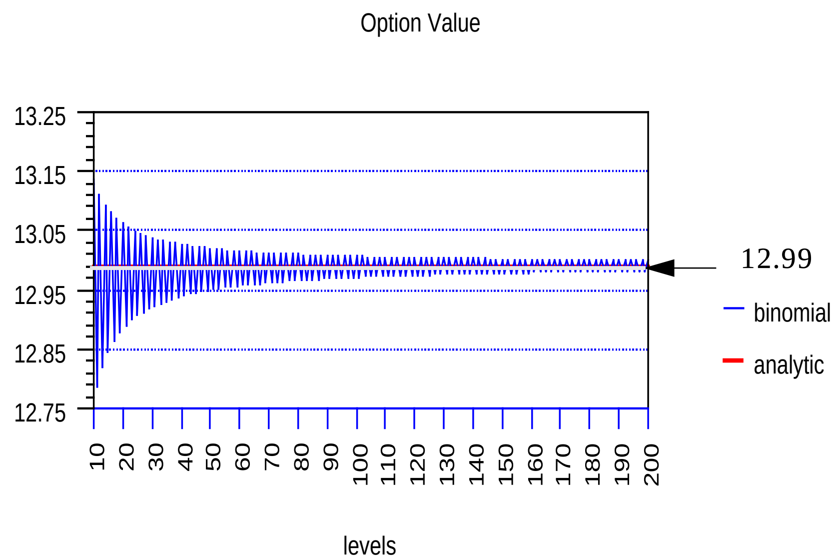
The binomial method for standard European-style options converges fairly rapidly as the number of levels on the binomial tree increases. Figure 1 shows the variation in value with level number for a typical case. You can see that the answer is accurate to better than 0.4% for binomial trees of greater than 40 levels; the values oscillate about the analytic value of 12.99 as you increment the number of levels, and approach the correct analytic value asymptotically.

In contrast, the binomial method for barrier options converges very slowly as the number of binomial levels increases, especially when the barrier is close to spot, as first pointed out by Margrabe (1989). Figure 2 illustrates the convergence of the binomial value of a representative down-and-out

1. For an overview of barrier options and their uses, see, for example, Derman and Kani (1993).

2. See Derman and Kani (1994), Rubinstein (1994), Dupire (1994).

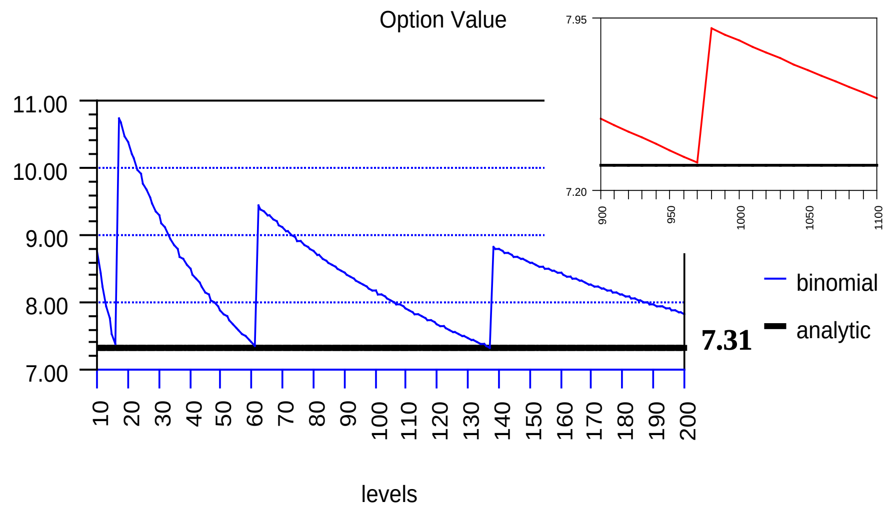
FIGURE 1. Convergence to analytic value of a binomially-valued one-year European call option as the number of binomial levels increases. We assume an index level of 100, a strike of 100, an annually compounded riskless interest rate of 10% per year, zero dividend yield, and a volatility of 20%. The analytic value is 12.99.



call option to its analytic value. The solution approaches the analytic value of 7.31 in a sawtooth fashion, with severe periodic spikes that move away from the correct result. The magnitude of the spikes attenuate so slowly with increasing periods that even between 900 and 1100 periods, as shown in the inset to Figure 2, the amplitude of the error due to the spike is 0.60, or about 8.2% of the correct theoretical option value. It may take tens of thousands of periods before the value converges to within 1%. Trinomial trees, implicit, explicit and other finite-difference methods suffer from similar problems.

In this paper we analyze the cause of this unsatisfactory convergence, and explain and illustrate a general method for improving it. Our method is applicable to all types of finite-difference methods.

FIGURE 2. Convergence to analytic value of a binomially-valued one-year European down-and-out call option as the number of binomial levels increases. We assume an index level of 100, a strike of 100, a barrier level of 95, an annually compounded riskless interest rate of 10% per year, zero dividend yield, and a volatility of 20%. The analytic value is 7.31. Inset shows convergence between 900 and 1100 binomial levels.



ERRORS ON A LATTICE

Options valuation often involves the solution of a boundary-value problem. You know the future payoffs of the option at its terminal boundaries, as dictated by the contract. The ability to hedge with the underlier dictates the replication strategy (and the corresponding continuous-time differential equation) that relates these future payoffs to the present fair value. To solve the differential equation numerically, you convert it to a finite difference equation on a discrete underlier value- and time-lattice, and then solve this equation. As you decrease the size of the lattice spacing, you get closer to the continuous-time result[3].

There are (at least) two sources of inaccuracy in modeling options on a lattice. In most of this paper we will illustrate our methods through the use of the generally familiar binomial tree for stock prices, but we stress that the same effects appear in any lattice scheme.

Stock Price Quantization Error

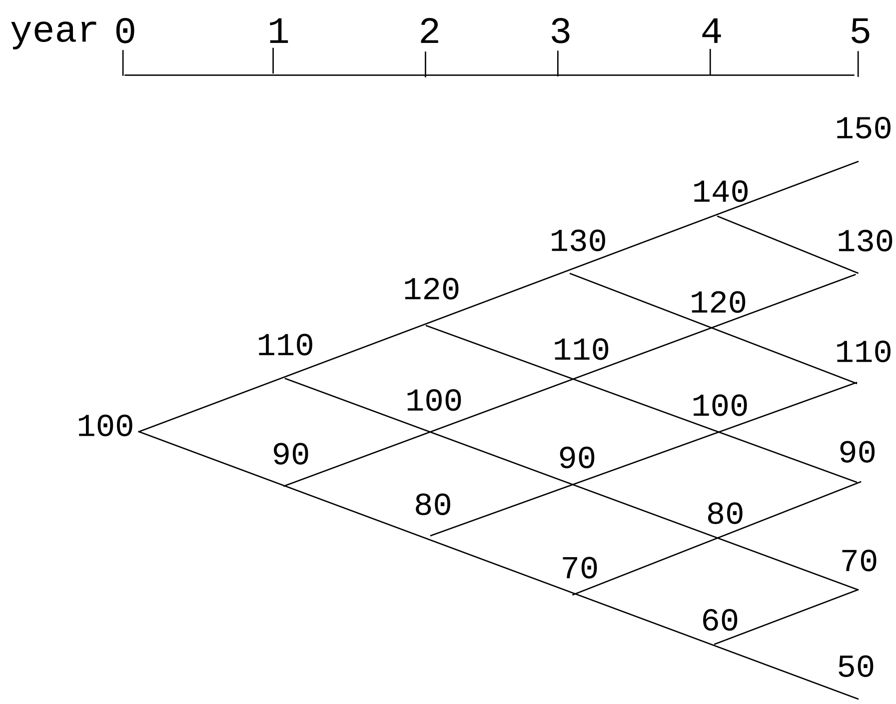
The first type of inaccuracy is caused by the unavoidable existence of the lattice itself, which “quantizes” the stock price and the instants in time at which it can be observed. Figure 3 contains a binomial lattice for a stock that moves up or down by $10 every year. We choose these coarse arithmetic (rather than geometric) increments to the stock price so as to keep the illustration simple rather than realistic. Once you’ve chosen a lattice, the stock is allowed to take the values of only those points on the lattice. In essence, when you use a lattice you are valuing an option on a stock that moves discretely. We call this unrealistic lack of continuity *quantization error.* It leads to an option price that is theoretically correct only for a stock that actually displays such quantized behavior; if you want to use the model for options on real stocks that move almost continuously, you must use a lattice with an infinitesimal mesh, or at least one small enough so that further reduction in its spacing has negligible numerical effect.

Option Specification Error

The second type of inaccuracy occurs because of the inability of the lattice to accurately represent the terms of the option. Once you’ve chosen a lattice, the available stock prices are fixed. If the exercise price or barrier level of the option doesn’t coincide with one of the available stock prices, you effectively have to move the exercise price

3. Real stocks trade at discrete times, with discrete ticks, pay discrete dividends, and trade only during certain periods. In some contracts barriers are only operative at certain times of day. In the interests of precise modeling, you may sometimes not want to proceed all the way to the continuous-time limit, but rather preserve the contract’s or market’s discreteness. We ignore these effects here, and assume that stock prices can move continuously.

FIGURE 3. A sample binomial stock lattice. The stock price (rather than its return) undergoes a simple arithmetic Brownian motion with a volatility of 10 points once a year. Although this is unrealistic, it makes the numerical examples simpler to explain, without giving up anything essential.



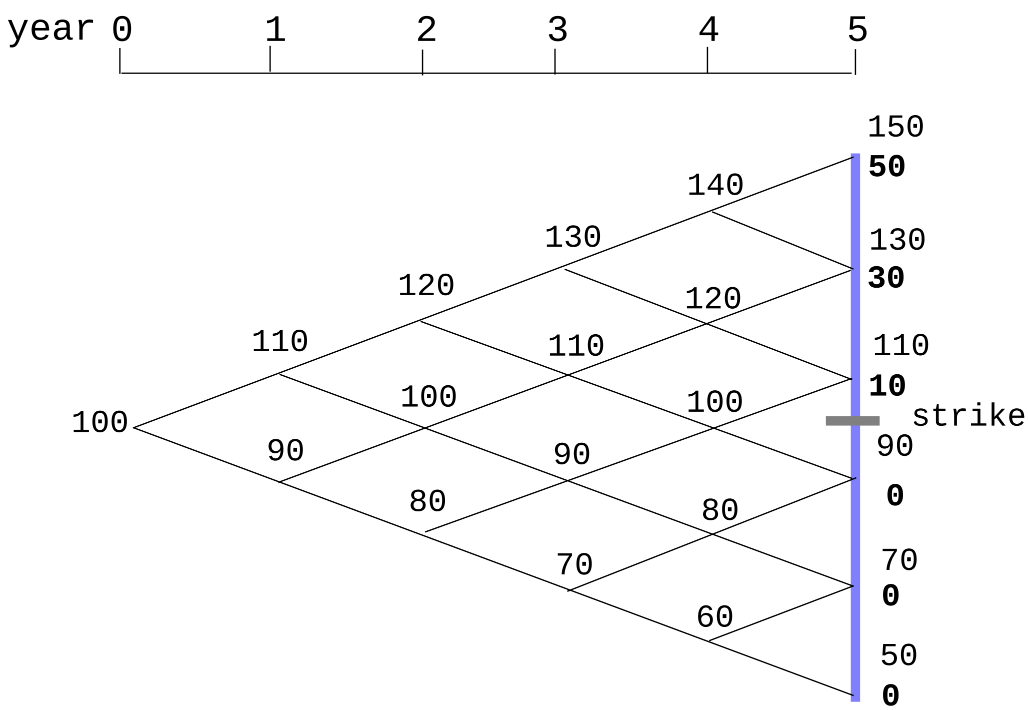
or barrier to the closest stock price available. Then, the option you value on the lattice has contractual terms that differ from those of the actual option. We call this *specification error*.

We will show that for barrier options, specification error vanishes much more slowly than quantization error. However, you can adjust lattice methods for specification error to get a much improved result.

Standard Options on a Lattice: Quantization Error

Consider a standard at-the-money call, struck at 100, with five years to expiration. Even on a coarse stock lattice you can choose a mesh so that a set of stock nodes coincide with the expiration time of five years. In Figure 4 we show the payoff of this call at the heavy expiration boundary on the binomial lattice of Figure 3. At all available nodes at expiration where the call's payoff is defined by the contract, the option's payoff is strictly correct. There is no *specification error*, only *quantization error*: only the movement of the stock is being mod-

FIGURE 4. The payoff of a five-year call struck at 100 on the binomial lattice of Figure 3. Call prices on the heavy expiration boundary are shown in bold type.

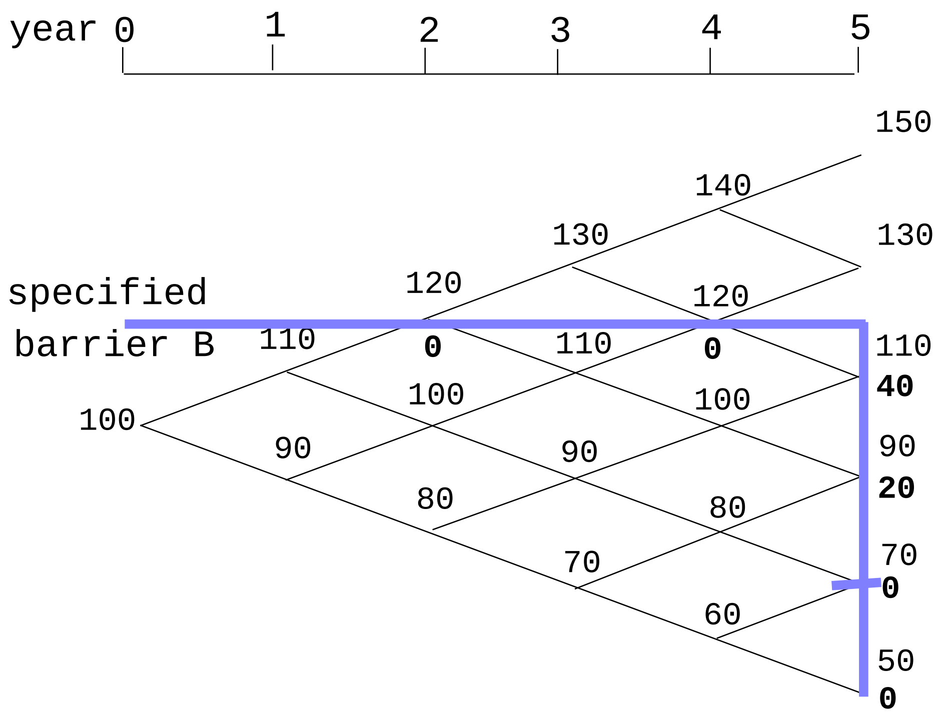


eled unrealistically. As long as the final nodes of the tree are placed at times corresponding to the expiration of the option, you are always valuing the correct option. As the number of binomial levels is increased, the quantization error diminishes. This behavior is manifested in Figure 1, where, in addition, because successive tree levels alternate between even and odd numbers of nodes, the quantization error about the analytic solution alternates in sign.

Barrier Options on a Lattice: Quantization and Specification Error

Let's look at an up-and-out European-style call option struck at 70, with five years to expiration and a knockout barrier at 120. Figure 5 shows the boundary values of this call on the sample binomial lattice of Figure 3. There are a set of tree nodes that lie exactly on the expiration boundary, so the payoff at each node is exactly the value dictated by the terms of the call contract. There are also a set of nodes that lie exactly on the knockout boundary at 120, so the payoff at each node on the knockout boundary is also correct. There is no specification error on the lattice; the only inaccuracy in the valuation of

FIGURE 5. The payoff of a five-year up-and-out call struck at 70 with knockout barrier at 120, illustrated on the binomial lattice of Figure 3. The specified expiration boundary and the barrier are denoted by a heavy line. Call payoffs on these boundaries are shown in bold type.

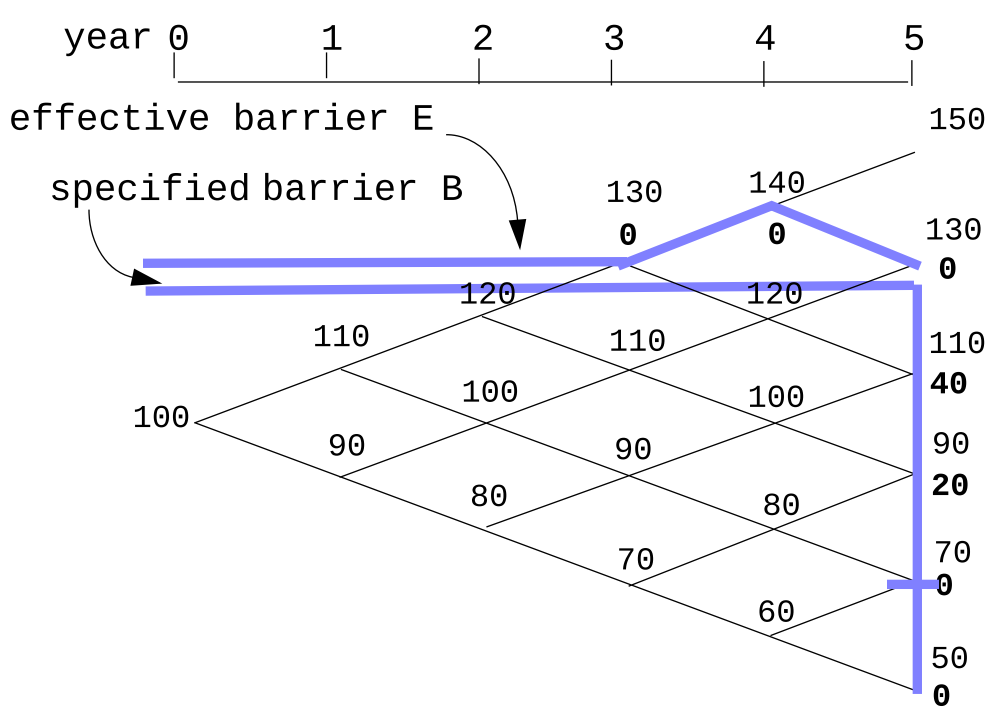


the call occurs from quantization error — the lattice mesh is too coarse to represent realistic stock price behavior. In the limit as the lattice mesh becomes infinitesimally small, the errors will vanish.

In contrast, Figure 6 shows a similar call with a knockout barrier at 125. Because the barrier falls between the nodes at 120 and 130, the lattice first “feels” the effect of the barrier at 130, in the sense that the nodes at 130 are the lowest-price nodes where the knockout boundary condition can be logically applied. So, call values at the nodes at 130 are set to zero. We call 130 the *effective barrier.*

If you now use this lattice to value the option, there are *two* ways in which you are valuing the wrong option. First, you are valuing an option whose barrier is really at 130 instead of 125. Second, as was the case with the standard option, the stock evolution is unrealistically coarse. As you decrease the lattice spacing, the effective barrier moves closer to the specified barrier — the specification error diminishes — and the stock evolution becomes more nearly continuous. From the point of view of accuracy, this situation is worse than the one for standard options, where the specification error was zero for any lattice spacing, and only the stock evolution was discontinuous. This is the reason for the worse convergence in Figure 2 compared with Figure 1.

FIGURE 6. The payoff of a five-year up-and-out call struck at 70 with knockout barrier at 125, illustrated on the binomial lattice of Figure 3. The specified barrier lies at 125. The effective barrier at which the lattice first perceives the effect of the knockout lies at 130.



ELIMINATING SPECIFICATION ERROR FOR BARRIER OPTIONS

Boyle and Lau (1994) have recently pointed out a method of improving binomial lattice valuation in certain cases. If you look at the saw-tooth pattern of convergence in Figure 2, you can see that for binomial trees with about 15, 60 or 138 levels, the binomial value is very close to the correct analytical value. The reason is that for these numbers of levels, the barrier falls almost exactly on the nodes and, in our language, the specification error is close to zero.

For a Cox-Ross-Rubinstein (1979) tree (a *CRR* tree) with N periods to expiration, a barrier at stock level B lies exactly m nodes away from the current stock price S when $S\exp\left(m\sigma\sqrt{\frac{T}{N}}\right) = B$ or

$$N = \left[(m^2\sigma^2 T)/\left|ln\frac{B}{S}\right|\right] \qquad m = \pm1, \pm2, \ldots \qquad \text{(EQ 1)}$$

This argument relies on the fact that the locations of the stock nodes of a CRR tree are independent of the riskless interest rate, and lie at the same stock levels at all times.

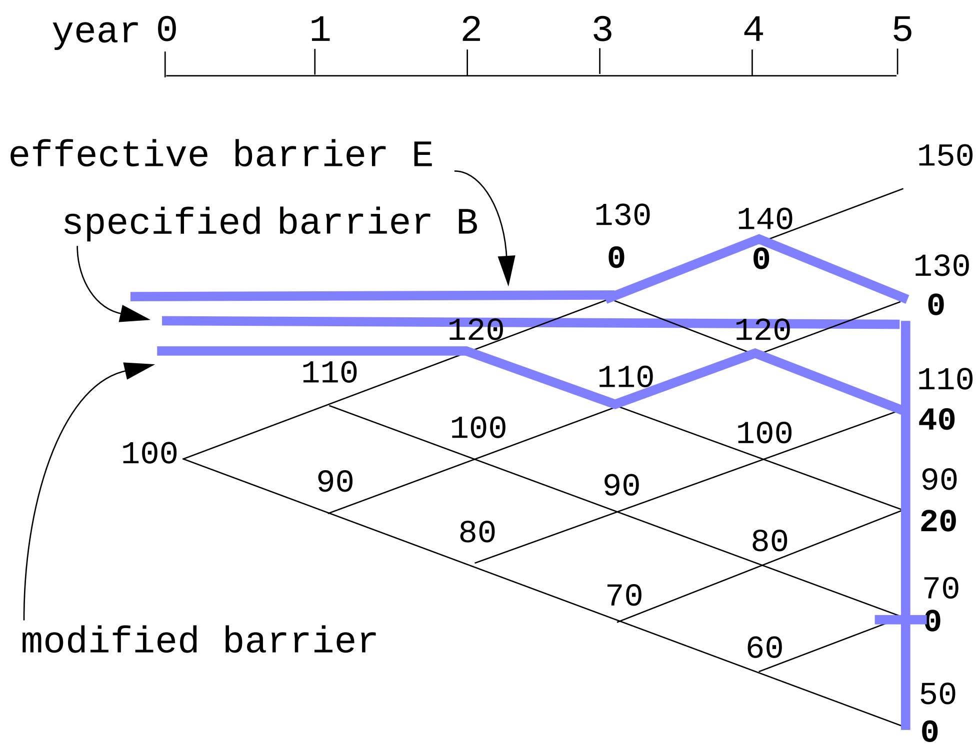
If the barrier level varies with time, or a different (non-CRR) tree is used, you cannot easily force the specification error to be zero by using Boyle and Lau's procedure. Therefore, we seek a method that corrects for the specification error no matter what the shape of the lattice or where the barrier falls relative to the lattice.

The Modified Barrier Method

We'll illustrate our strategy by referring to the option in Figure 6, in which the effective barrier lies at 130 but the (true) specified barrier lies at 125. When you value the option on this tree, the computed option values at the first set of tree nodes just inside the specified barrier will be incorrect, because they have been naively computed from a knockout at the effective barrier rather than at the specified barrier. We call this first set of nodes with computed values the *modified barrier*, and display it in Figure 4. The values on these modified barrier nodes obtained by valuing the option using backward induction from the effective barrier are larger than they should be, because the contract dictates that the call knocks out at 125, and the binomial lattice first "feels" the knockout at 130.

Therefore, in the interests of accuracy, we must adjust the naively-computed values at the modified barrier nodes. We will replace them by values that more accurately reflect their closer proximity to the knockout barrier at the specified level of 125. After modifying the val-

FIGURE 7. The *modified barrier* for the knockout option of Figure 6. Stock prices are shown in plain type. Call payoffs are shown in bold.



ues at the modified barrier, we will continue with valuation by backward induction towards the root of the tree, and so obtain the current option value.

What's the right way to modify the naively-computed values on the modified barrier? First, notice that the naive values at all tree nodes *are* appropriate for an option contract with barrier level at the effective barrier E. Their only inaccuracy is due to quantization error. Therefore, you can use this naive tree with knockout occurring at the effective barrier to compute a reasonably accurate finite-difference approximation for the $\left[\frac{\partial C}{\partial S}(S, E, t)\right]_{S=E}$. This is the rate at which the barrier option's value C varies with stock price S near its effective barrier E at all future times t. This derivative is also a good approximation for the rate at which the value of the option, with knockout occurring at the specified barrier B, grows away from the barrier.

Because the distance between the specified barrier B and the effective barrier E is small, the rate at which a barrier option value grows away from the barrier is independent of the location of the barrier to first order, that is

$$\left[\frac{\partial C}{\partial S}(S, E, t)\right]_{S = E} = \left[\frac{\partial C}{\partial S}(S, B, t)\right]_{S = B} + O((B - E)^2) \qquad \textbf{(EQ 2)}$$

Equation 2 provides an estimate for the derivative, at the barrier B, of the value of the option with knockout occurring at barrier B. We can use this derivative to develop a first-order Taylor expansion for the option value about its specified barrier, and so get a more accurate value of the option on its modified barrier. We can then value the option by backward recursion from this modified barrier to find a more accurate solution.

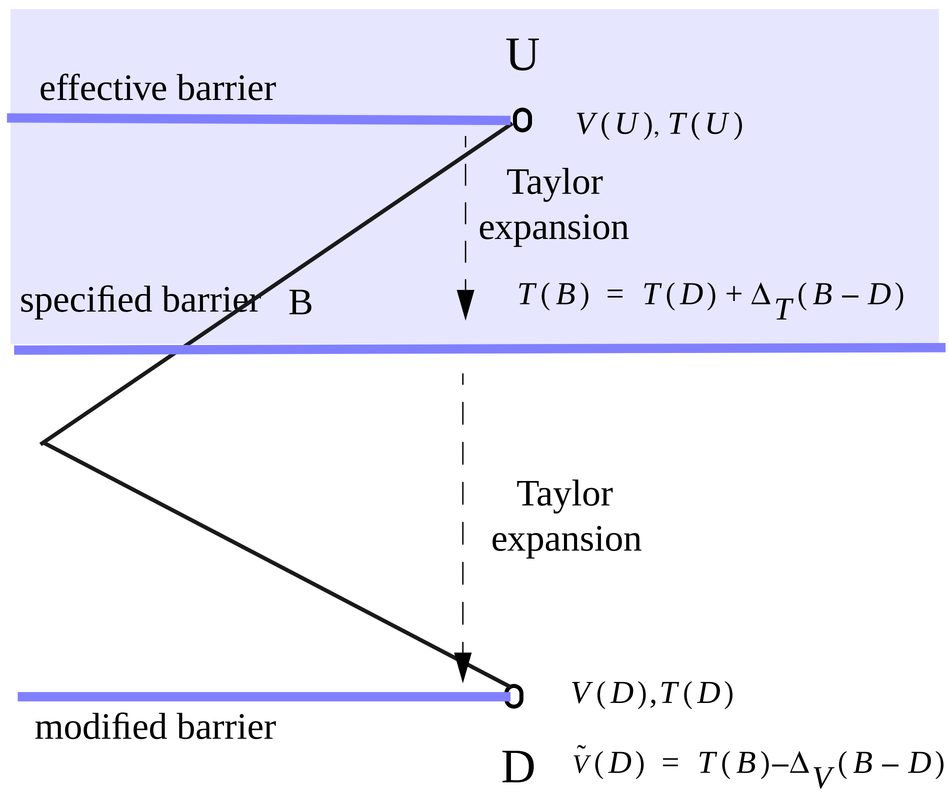
That's the procedure for enhancing the value of a knockout option. If the option C knocks *into* a target option T on a barrier B that lies between tree levels, we can use the same reasoning to obtain the estimated value of T on the specified barrier. Use the naive tree with the effective barrier to get an estimate for $\left[\frac{\partial T}{\partial S}(S, E, t)\right]_{S = E}$. Then use this derivative to develop a first-order Taylor expansion that computes the value of *T(B)* on the specified barrier from its zeroth-order value on the modified barrier. This value of *T(B)* provides an enhanced estimate for the value of the target option on the specified barrier. This value is then used as the zeroth-order term in a first-order Taylor expansion for the value of the barrier option on the modified barrier. The method is illustrated in detail in A BINOMIAL EXAMPLE on page 17.

To summarize, the modified barrier method is a sort of bootstrap method. You first value the (slightly) *wrong* option by backward induction from the *wrong* (effective) barrier to get (almost) *right* numerical values for the derivative of the *true* option at all times on its barrier. You then use these derivatives at each level of the tree in a first-order Taylor series on the barrier to obtain modified barrier values for the true option. Finally, you value the correct option by backward induction from the modified barrier.

The Modified Barrier Algorithm

Consider an option with value *V(S)* that knocks into a target security *T(S)* if the stock price *S* crosses a barrier *B*. (You can think of *T(S)* as being zero for a knockout option that pays no rebate.) Figure 8 shows the method we use to correct the values on the modified barrier.

FIGURE 8. The modified barrier algorithm on a binomial tree. U is the up-node on the effective barrier above the specified barrier. *B* represents the specified barrier that in general falls between an up- and a down-node. D is the down-node on the modified barrier below the specified barrier. *V(S)* is the value of the barrier option at stock price *S*, assuming the specified barrier coincides with the effective barrier. *T(S)* is the value of the target option which *V(S)* knocks into on the effective barrier. $\tilde{V}(D)$ is the adjusted value of the barrier option on the modified barrier.



1. Value the target option *T(S)* and the barrier option *V(S)* at each node on the tree with the barrier at the effective barrier.

2. Calculate the finite-difference derivatives with respect to stock price for each option on the effective barrier:

$$\Delta_T = \frac{T(U) - T(D)}{U - D}$$
$$\Delta_V = \frac{V(U) - V(D)}{U - D} \quad \text{(EQ 3)}$$

3. Use a first-order finite-difference Taylor series for *T()* to calculate the value of the target option on the specified barrier *B* from its value at node *U:*

$$T(B) = T(U) + \Delta_T(B - D) \quad \text{(EQ 4)}$$

4. Use a similar Taylor series for *V()* to calculate the corrected value of the barrier option on the modified-barrier node *D* from its knock-in value *T(B)* on the specified barrier:

$$\tilde{V}(D) = T(B) - \Delta_V(B - D) \quad \text{(EQ 5)}$$

5. Use backward induction from the modified barrier with $\tilde{V}(D)$ as the nodal boundary values to find the value of *V(S)* at all other nodes inside the barrier.

Another Interpretation: The Modified Barrier Algorithm as Interpolation at the Barrier

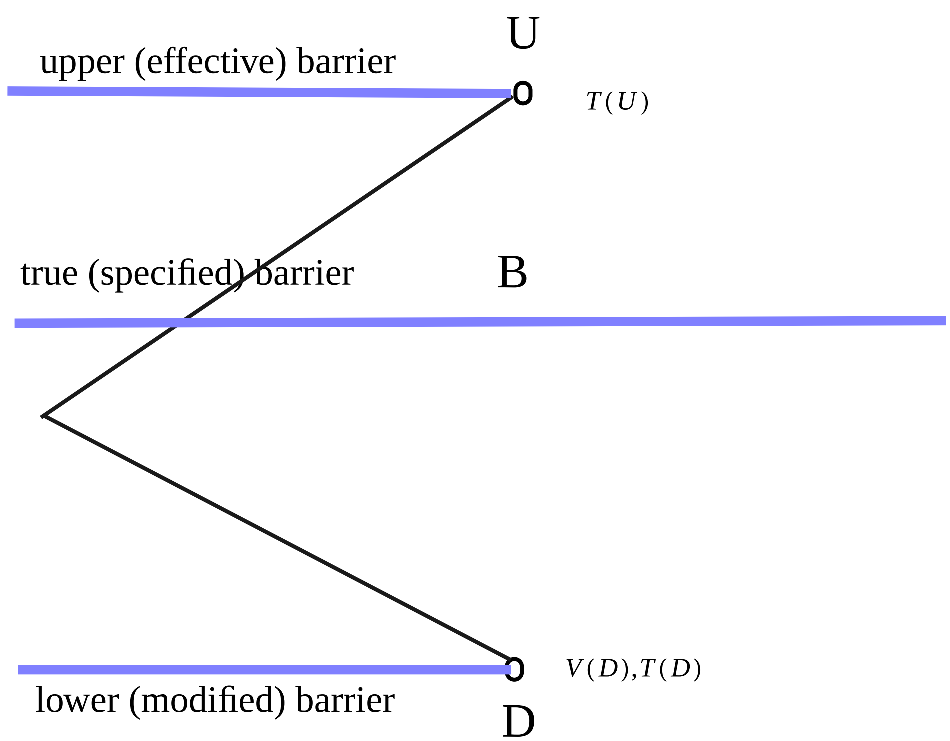
There is another, more intuitive way to understand this modified barrier expansion. Because the specified barrier lies between two sets of nodes on the tree, it is tempting to regard the correct option value as the one obtained by interpolating the two option values corresponding to 1) moving the barrier up to the effective barrier and 2) moving the barrier down to the modified barrier. In fact, the algorithm described in the previous section is equivalent to this procedure, provided the interpolation at the barrier is done at every time period on the tree or lattice.

Here's the algorithm from this point of view, described with reference to Figure 9:

1. Value the target option *T(S)* and the barrier option *V(S)* with the barrier moved up to the effective barrier. In Figure 9 we call this the upper barrier. The computed value of *V(S)* on this modified barrier is then *V(D)*, the value obtained from an unenhanced calculation.

2. Similarly, value *T(S)* and *V(S)* with the specified barrier moved down to the modified barrier. In Figure 9 we call this the lower barrier. The value of *V(S)* on the modified barrier is then precisely *T(D)*, the value of the target option it knocks into.

FIGURE 9. The modified barrier algorithm interpreted as an interpolation between the two barriers. When the specified barrier is moved up to the effective barrier at the level of node *U*, the barrier option at node *D* has value *V(D)*. When the specified barrier is moved down to the modified barrier at the level of node *D*, the barrier option at node *D* has value *T(D)*, the value of the target option it knocks into at that level. $\tilde{V}(D)$ is the interpolated value of the barrier option on the modified barrier.



$$\tilde{V}(D) = \left(\frac{B-D}{U-D}\right)V(D) + \left(\frac{U-B}{U-D}\right)T(D)$$

3. Replace *V(D)* on the lower barrier by the value $\tilde{V}(D)$ obtained from interpolating between *V(D)* and *T(D)* according to *B's* distance from the effective barrier and the modified barrier:

$$\tilde{V}(D) = \left(\frac{B-D}{U-D}\right)V(D) + \left(\frac{U-B}{U-D}\right)T(D) \qquad \text{(EQ 6)}$$

4. Use backward induction from the modified barrier with $\tilde{V}(D)$ as the boundary values to find the value of *V(S)* at all other nodes inside the barrier.

Equivalence of the Interpolation Method and the Taylor Expansion Method

The formula resulting from interpolation at the barrier is equivalent to the formula resulting from the Taylor expansion. You can show that the formulas in Equation 5 and Equation 6 are identical by substituting Equation 3 and Equation 4 into Equation 5 to obtain Equation 6.

In-Out Parity

A long position in an in-option and an out-option with the same barrier provides the same payoff as a long position in the same type of option with no barrier. Any model for barrier options must satisfy this in-out parity relationship.

The values that results from applying our enhancement algorithm to the binomial model preserve this relation, provided the unenhanced binomial values from which they start satisfy it.

We'll illustrate this for barrier options that knock into and knock out of a target call option *C*. Let *C(D)* denote the unenhanced value of a call on the modified barrier *D*, and let *I(D)* and *O(D)* denote the unenhanced values of an in- and an out- call at the same node. In the standard binomial model,

$$C(D) = I(D) + O(D) \quad \text{(EQ 7)}$$

at all nodes. It is true on the barrier by specification, and true at earlier times by backward induction.

From Equation 6, the value of the out-call on the modified barrier after enhancement is

$$\begin{aligned}\tilde{O}(D) &= \left(\frac{B-D}{U-D}\right)O(D) + \left(\frac{U-B}{U-D}\right)T(D) \\ &= \left(\frac{B-D}{U-D}\right)O(D) + 0\end{aligned} \quad \text{(EQ 8)}$$

where *T(D)* is the target boundary value of the out-option on the barrier, and equals zero for a knockout call.

Similarly, the value of the in-call after enhancement is

$$\begin{aligned}\tilde{I}(D) &= \left(\frac{B-D}{U-D}\right)I(D) + \left(\frac{U-B}{U-D}\right)T'(D) \\ &= \left(\frac{B-D}{U-D}\right)O(D) + \frac{(U-B)}{(U-D)}C(D)\end{aligned} \quad \text{(EQ 9)}$$

where $T'(D)$ is the target boundary value of the in-option on the barrier, and equals the value $C(D)$ of the call itself.

Adding the above two equations gives

$$\begin{aligned}\tilde{O}(D) + \tilde{I}(D) &= \frac{(B-D)}{(U-D)}[O(D) + I(D)] + \left(\frac{U-B}{U-D}\right)C(D) \\ &= \frac{(B-D)}{(U-D)}[C(D)] + \left(\frac{U-B}{U-D}\right)C(D) \\ &= C(D)\end{aligned} \quad \text{(EQ 10)}$$

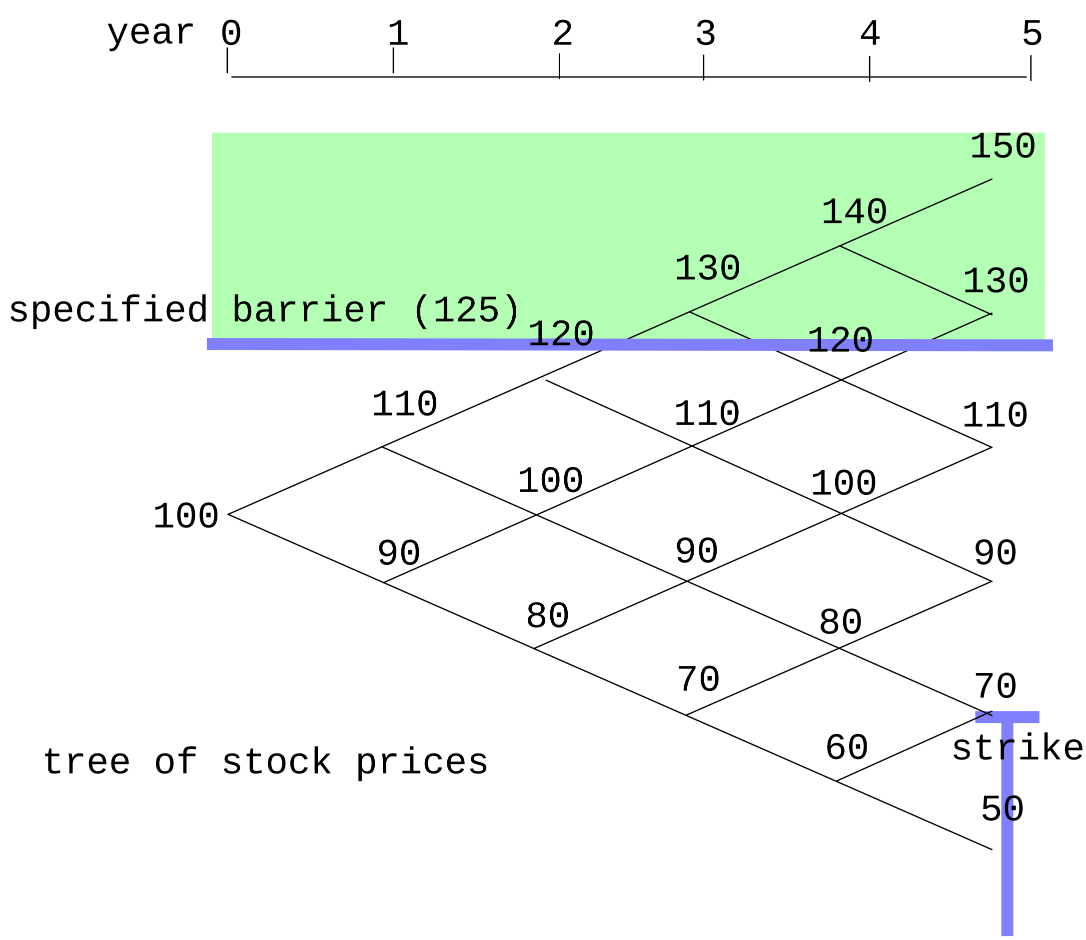
where the second line of Equation 10 follows from the unenhanced form of in-out parity in Equation 7, and the last line follows from simple algebra. The enhanced binomial values satisfy in-out parity.

A BINOMIAL EXAMPLE

In this section we illustrate how to implement the method on a simple binomial tree.

Figure 10 contains the stock tree of Figure 3. It corresponds to a normal stock price volatility of 10 points per year, with zero interest rates and zero dividend yields. Also shown are the expiration boundary for an up-and-out five-year call on the stock with a strike of 70 and a knockout barrier at 125.

FIGURE 10. **An up-and-out call with strike of 70 and out-barrier of 125 on a sample binomial tree. The stock is assumed to have a normal price volatility of 10 points per year, with moves occurring only once per year. We also assume zero dividend yield and zero interest rates. All transition probabilities on the tree are identical and equal to 1/2.**



The first tree in Figure 11 shows the unenhanced valuation of the up-and-out call when the barrier is moved up to the effective barrier, according to the first step in our algorithm. The value at each node in year 5 is the value of the call at expiration. At stock levels of 130 or higher, the call is knocked-out and therefore worth zero at all nodes on the effective barrier. The option value C_n at any node n inside the effective barrier and the expiration boundary is computed from the values C_u and C_d using the usual binomial discounted expectations formula with equal probabilities and zero interest rates:

$$C_n = \frac{C_u + C_d}{2} \qquad \textbf{(EQ 11)}$$

The current unenhanced value of the call at the root of the tree is found to be 17.50.

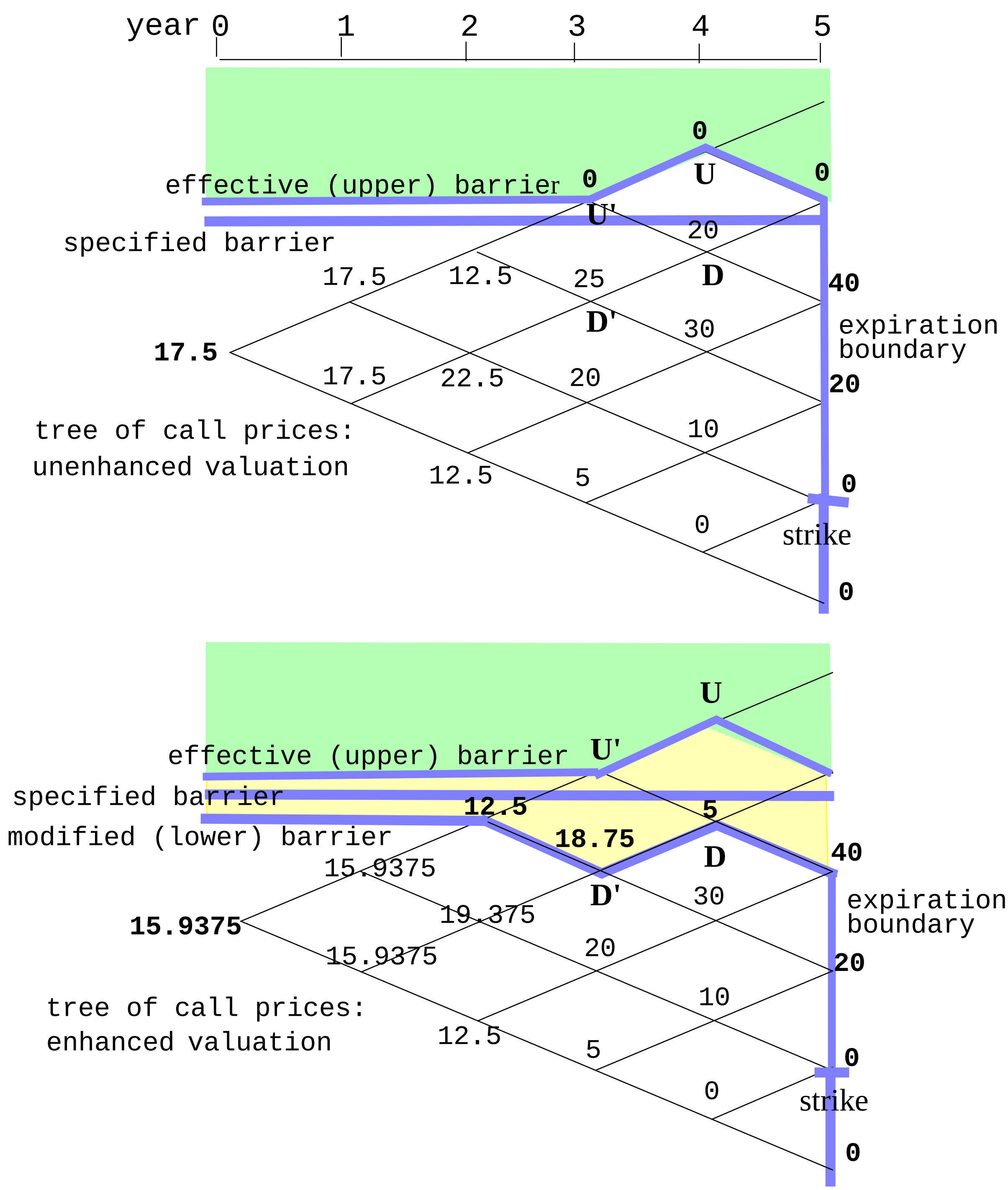
Now let's correct the values on the modified barrier to allow for the fact that the specified barrier is actually closer than the effective barrier. In the lower tree of Figure 11, in year 4, the real barrier B = 125 lies between node U =140 on the effective barrier and node D = 120 on the modified barrier. The value of the knockout call at node D when the barrier is at the effective barrier is $V(D) = 20$; the value of the knockout call when the barrier is at the modified barrier is 0 (because it knocks out there). In the notation of the interpolation formula of Equation 6, the enhanced value of the call option at node D is given by

$$\begin{aligned} \tilde{V}(D) &= \frac{(125-120)}{(140-120)} \times 20 + \frac{(140-125)}{(140-120)} \times 0 \\ &= \frac{5}{20} \times 20 + \frac{15}{20} \times 0 \\ &= 5 \end{aligned} \qquad \textbf{(EQ 12)}$$

The enhanced value at node D in the second tree of Figure 11 is 5, lower than the unenhanced value of 20, because the true barrier is closer than the effective barrier.

We can use the same formula for interpolating between the $U' = 130$ and $D' = 110$ nodes in year 3 of Figure 11. The unenhanced value of the call at node D' in the upper tree is $V(D')$ = 25. The enhanced value is given by

FIGURE 11. **Valuing the up-and-out call on the tree of Figure 4.**



$$
\begin{aligned}
\tilde{V}(D) &= \frac{(125-110)}{(130-110)} \times 25 + \frac{(130-125)}{(130-110)} \times 0 \\
&= \frac{15}{20} \times 25 + \frac{5}{20} \times 0 \\
&= 18.75
\end{aligned}
\qquad \text{(EQ 13)}
$$

This value is again lower than the unenhanced value, because the D' node lies closer to the true barrier than to the effective barrier.

The lower tree of Figure 11 now has payoff values for the up-and-out call on both the expiration boundary and the modified barrier. We can use the backward induction formula of Equation 11 to calculate the option values at all nodes interior to the boundaries. The current enhanced option value at the root of the tree is found to be 15.94, lower than the unenhanced value of 17.50 found previously.

When interest rates and dividend yields are non-zero, you can use exactly the same methodology to diminish the specification error and so get enhanced values for barrier options. In practice you would need to use trees with about 100 levels rather than 5 or 6.

THE EFFICACY OF THE ENHANCEMENT

We now present several examples that illustrate the improvements obtained in using this method, both on binomial trees and on more general lattices.

In Figure 2 we showed the slow convergence of the unenhanced binomial method for a one-year European down-and-out call option, with strike at 100 and barrier at 95, as a function of the number of tree levels. Figure 12 shows the improved convergence of the enhanced values. You can see how much more rapidly the enhanced values approach the analytical result as the number of levels increase. The sawtooth behavior damps out at a faster rate. For numbers of levels greater than 80, the result is virtually perfect.

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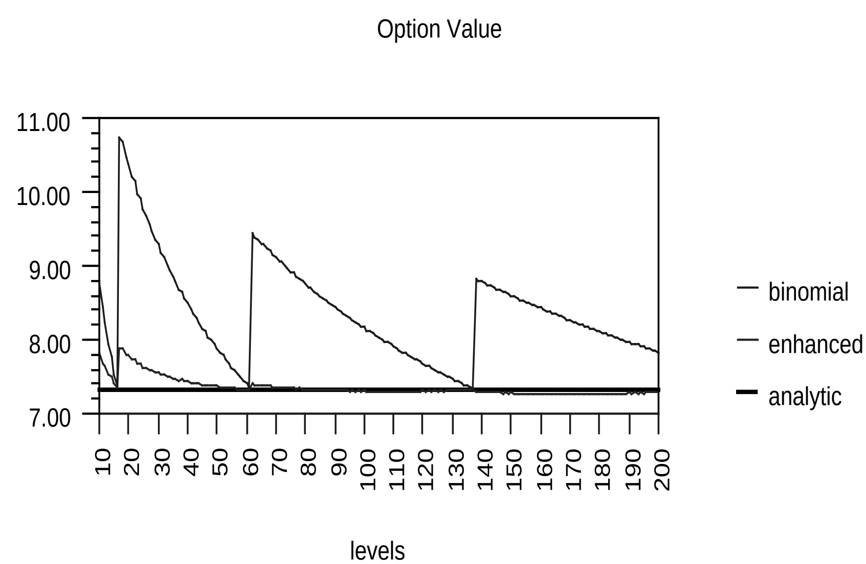
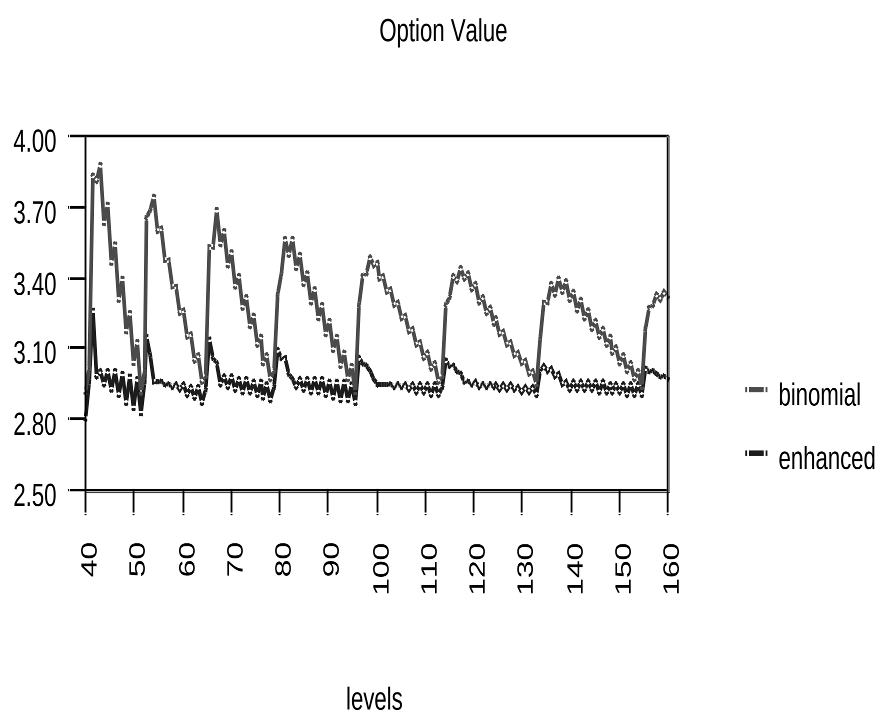
FIGURE 12. Convergence to analytic value of an enhanced binomially-valued one-year European down-and-out call option as the number of binomial levels increases. We assume an index level of 100, a strike of 100, a barrier level of 95, an annually compounded riskless interest rate of 10% per year, zero dividend yield, and a volatility of 20%. The analytic value is 7.31.



In the above example we knew the exact analytic value for the option, so the enhancement was not really necessary. Now let's look at some other examples where the enhancement is important because the analytic solution is unavailable.

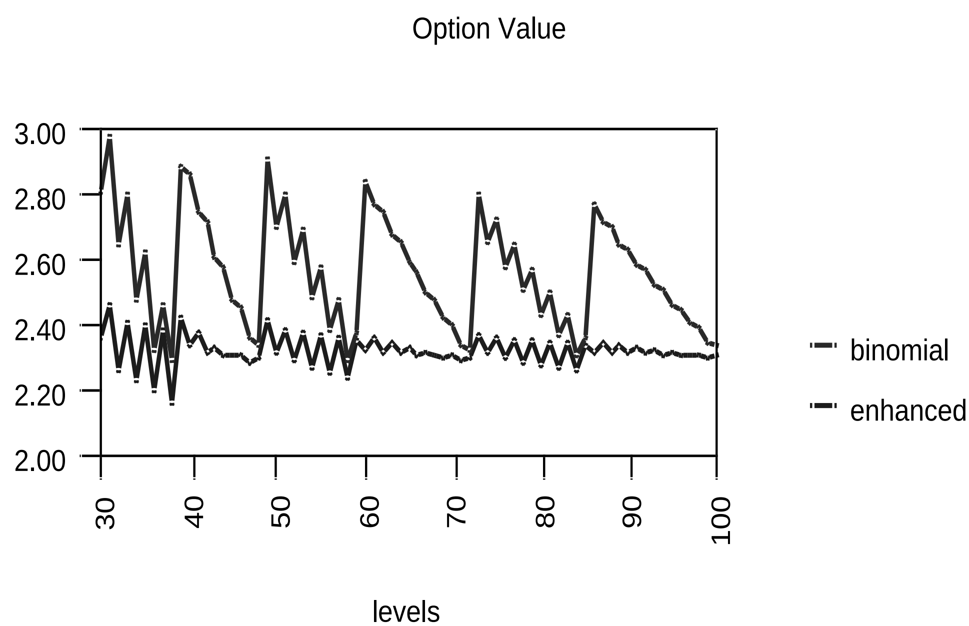
Figure 13 demonstrates the enhancement obtained for an up-and-out call with the same interest rates and volatility, but with a barrier that increases linearly through time from a level of 120 at the start of the option's life to a level of 130 at expiration. Again the enhanced solution has a much smaller sawtooth amplitude than the plain binomial method. Between 90 and 100 tree levels, the magnitude of the sawtooth is about five times smaller for the enhanced solution than for the plain binomial solution. There is no known analytic solution for this type of barrier, so that an enhanced method saves both computing time and provides greater accuracy.

FIGURE 13. Convergence of an enhanced binomially-valued one-year European up-and-out call option as the number of binomial levels increases. We assume an index level of 100, a strike of 100, an annually compounded riskless interest rate of 10% per year, zero dividend yield, and a volatility of 20%. The barrier increases linearly with time from 120 to 130.



Barrier options valued on an implied tree[4] in the presence of a general volatility smile also have no analytic solution. Figure 14 illustrates the enhancement obtained for an up-and-out call with barrier at 120, but with a volatility skew for the underlying stock. We assume a skew that varies linearly from an implied volatility of 20% for out-of-the-money puts struck at 60 to an implied volatility of 12% for out-of-the-money calls struck at 140. Once again, the enhanced solution seems to be converging to the asymptotically correct value with much smaller sawtooth fluctuations.

FIGURE 14. Convergence of an enhanced binomially-valued one-year European down-and-out call option as the number of binomial levels increases. We assume an index level of 100, a strike of 100, a barrier level of 120, an annually compounded riskless interest rate of 10% per year, zero dividend yield, and a volatility skew that varies linearly from 20%for puts struck at 60 to 12% for calls struck at 140.

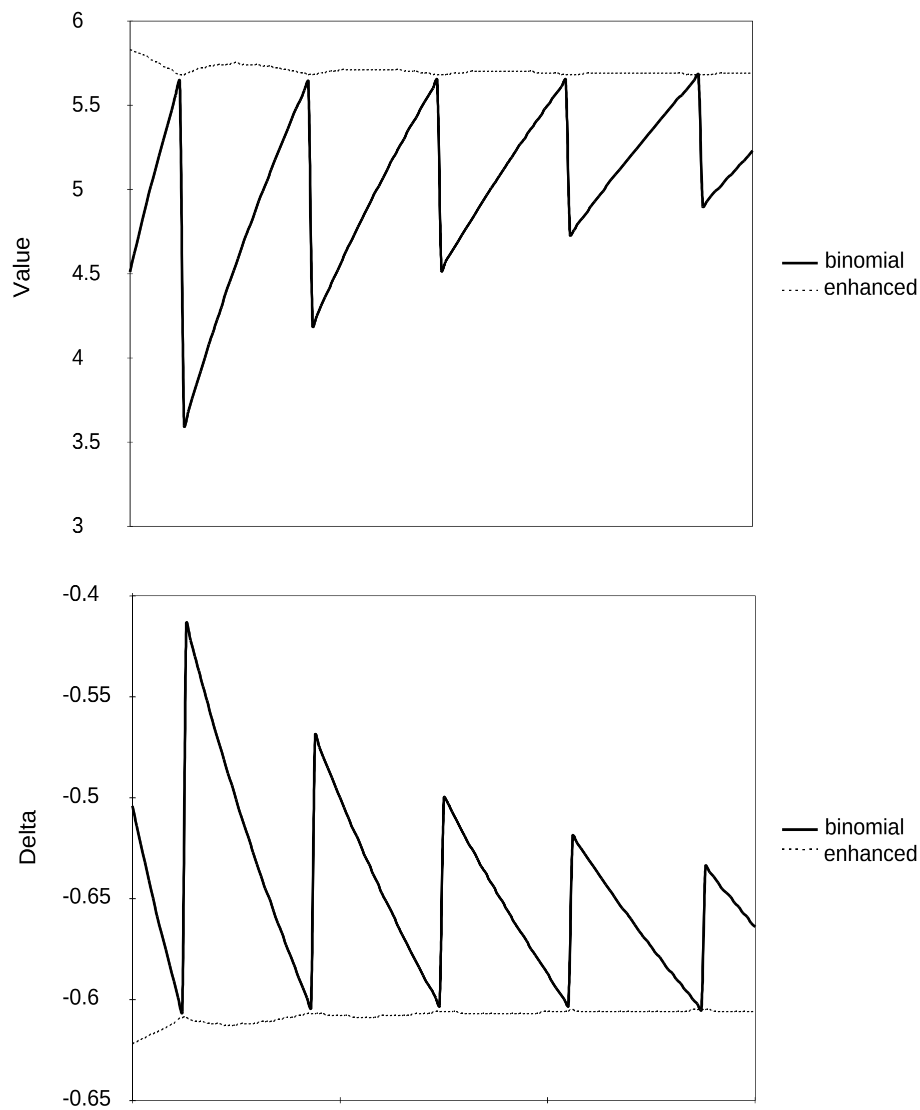


Finally, we stress that this enhancement method works for well-known lattice-based finite difference methods as well as the binomial method. Figure 15 shows the effect of our enhancement algorithm on the value and delta of a down-and-in call with strike at 100 and barrier at 95, valued using the implicit (trinomial) finite-difference

4. See Derman and Kani (1994).

method. The enhancement again leads to a much smoother and more rapidly converging option value and delta.

FIGURE 15. **Convergence of an enhanced binomially-valued one-year European down-and-in call option as the number of lattice levels along the stock axis increases. We assume an index level of 100, a strike of 100, a barrier level of 95, an annually compounded riskless interest rate of 10% per year, zero dividend yield, and a volatility of 20%.**



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